

# Spin squeezing: transforming one-axis-twisting into two-axis-twisting

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Squeezed spin states possess unique quantum correlation or entanglement that are of significant promises for advancing quantum information processing and quantum metrology. In recent back to back publications [C. Gross *et al*, *Nature* **464**, 1165 (2010) and Max F. Riedel *et al*, *Nature* **464**, 1170 (2010)], reduced spin fluctuations are observed leading to spin squeezing at  $-8.2\text{dB}$  and  $-2.5\text{dB}$  respectively in two-component atomic condensates exhibiting one-axis-twisting interactions (OAT). The noise reduction limit for the OAT interaction scales as  $\propto 1/N^{2/3}$ , which for a condensate with  $N \sim 10^3$  atoms, is about 100 times below standard quantum limit. We present a scheme using repeated Rabi pulses capable of transforming the OAT spin squeezing into the two-axis-twisting type, leading to Heisenberg limited noise reduction  $\propto 1/N$ , or an extra 10-fold improvement for  $N \sim 10^3$ .

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Squeezed spin states (SSS) [2, 3] are entangled quantum states of a collection of spins in which the correlations among individual spins reduce quantum uncertainty of a particular spin component below the classical limit for uncorrelated particles [2]. Research in SSS is a topical area due to its significant applications in high-precision measurements [3–10] and in quantum information science [11–15]. Squeezed spin states were first introduced by Kitagawa and Ueda, who considered two ways to produce them. The simplest to implement uses a “one-axis twisting” (OAT) Hamiltonian, but the state it produces does not have ideal squeezing properties. A more complex approach uses a “two-axis twisting” (TAT) Hamiltonian and produces an improved state. Other mechanisms for producing SSS have also been investigated, especially those based on atom-photon interactions [16, 17] and quantum non-demolition measurements [18–21].

Atomic Bose-Einstein condensates are promising systems for observing spin squeezing. Assuming fixed spatial modes, condensed atoms are described by a collection of pseudo-spin 1/2 atoms, with spin up ( $|\uparrow\rangle$ ) and down ( $|\downarrow\rangle$ ) denoting the two internal states or spatial modes [12, 22–25]. The two recent experiments [4, 5] raise significant hope for reaching the theoretical limit of spin squeezing  $\propto 1/N^{2/3}$  with  $N$  the total number of atoms for the OAT model [2]. Both experiments utilize two internal hyperfine states of condensed atoms, with the OAT interaction cleverly constructed from binary atomic collisions, possibly accompanied by systematic and fundamental imperfections not confined to the two state/mode approximation. They can be further degraded by atomic decoherence and dissipation [4, 5].

This Letter describes a readily implementable idea for improved spin squeezing in the two experiments. Given the reported OAT model parameters [4, 5], we propose

a coherent control scheme capable of transforming the OAT into the effective TAT spin squeezing, leading to a Heisenberg limited noise reduction  $\propto 1/N$ , or a further 10 fold improvement for a condensate with  $\sim 10^3$  atoms.

With a coherent coupling, the two-component condensate is described by the following Hamiltonian

$$H = \chi J_z^2 + \Omega(t) J_y, \quad (1)$$

as in [4, 5]. The collective spin  $\vec{J}$  is defined according to  $J_v = \sum_k \hat{\sigma}_v^{(k)}/2$  in terms of the Pauli matrices  $\hat{\sigma}_v^{(k)}$  ( $v = x, y, z$ ) for the pseudo-spin of the  $k$ -th atom. The first term on the right-hand side of Eq. (1) is the nonlinear interaction responsible for the OAT-type spin squeezing, with  $\chi$  the atomic interaction parameter. The Rabi frequency  $\Omega(t) = \Omega_0 f(t)$  results from near-resonant two-photon (microwave+rf) coupling between the hyperfine states [4, 5]. Its maximum amplitude  $\Omega_0$  and the temporal envelop  $f(t)$  can be controlled experimentally [4, 5].

Our idea for transforming the OAT (1) into the TAT makes use of multiple  $\pi/2$  pulses affected with the coupling term  $\Omega J_y$ . In the Rabi limit,  $|\Omega| \gg N|\chi|$ , nonlinear interaction can be neglected while the collective spin undergoes driven Rabi oscillation. A pulse with an area of  $\pi/2$  corresponds to  $\int_{-\infty}^{\infty} \Omega(t) dt = \pi/2$ , which gives the transformation

$$R_{-\pi/2} e^{iJ_z^2} R_{\pi/2} = e^{iJ_x^2}, \quad (2)$$

where  $R_\theta = e^{-i\theta J_y}$ , rotating the spin around the  $y$ -axis by an angle  $\theta$ . The proposed pulse sequence is periodic as shown in Fig. 1(a). Each period [Fig. 1(b)] is made up of the following: a  $\pi/2$  rotation about  $y$ -axis (red pulse), a free evolution for  $2\delta t$ , a second  $-\pi/2$  rotation about  $y$ -axis (blue pulse), and a second free evolution for  $\delta t$ . The period is  $t_c \approx 3\delta t$ , neglecting the time needed for affecting the two  $\pm\pi/2$  pulses.

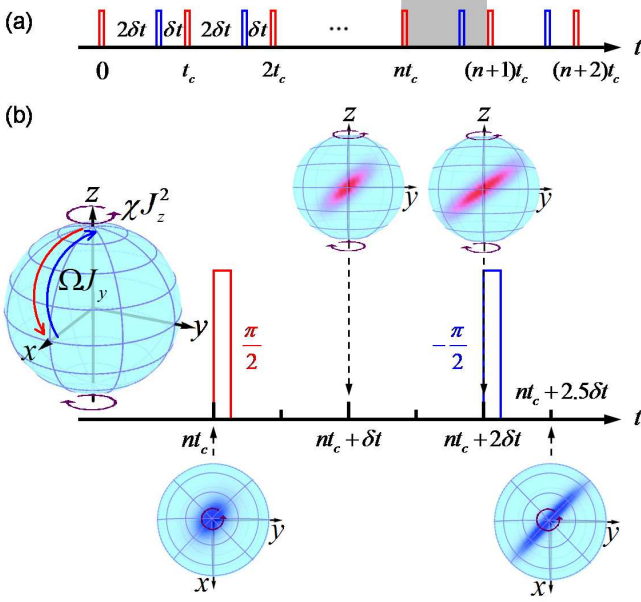


FIG. 1: An illustration of the proposed pulse sequence with pulse envelop  $\Omega/\Omega_0$  vs. time  $t$  in arbitrary units, and the associated spin distributions on Bloch spheres. (a) The multi-pulse sequence is cyclic, with one period between  $nt_c$  and  $(n+1)t_c$  (shaded) and zoomed in (b). Each period contains two  $\pi/2$  pulses. A red pulse rotates the spin from along the positive  $z$ -axis to the  $x$ -axis, and a blue one coherently phased to achieve the opposite, is a  $-\pi/2$  pulse, as shown on the large Bloch sphere of (b). The four small Bloch spheres are located at their corresponding times. The spin distributions for the upper two are centered on the  $x$ -axis, while the lower two are centered on the  $z$ -axis, due to the applications of red and blue  $\pi/2$  pulses. The twistings from the OAT interaction (1) are illustrated with counter-rotating circular arrows pierced by the  $x$ - or  $z$ -axis of all Bloch spheres.

The time evolution operator at time  $t = nt_c$  ( $n = 1, 2, 3, \dots$ ), i.e., after  $n$  periods, is given by  $U^n$ , where

$$U = e^{-i\tau J_z^2} R_{-\pi/2} e^{-2i\tau J_z^2} R_{\pi/2} = e^{-i\tau J_z^2} e^{-2i\tau J_z^2}, \quad (3)$$

is for a single period and  $\tau = \chi\delta t$ . Using Baker-Campbell-Hausdorff formula, we find

$$U = e^{-i\tau(2J_x^2 + J_z^2)} \exp[i\tau^2\{J_x, \{J_y, J_z\}\} + \mathcal{O}(\tau^2)],$$

where  $\{A, B\} = AB + BA$  denotes anti-commutator for operators  $A$  and  $B$ . Expanding for small  $\tau$ , we arrive at  $\exp[i\tau^2\{J_x, \{J_y, J_z\}\} + \mathcal{O}(\tau^2)] \simeq 1$  after neglecting higher order terms. Hence we end up with  $U \simeq e^{-i\tau(2J_x^2 + J_z^2)}$ , and the time evolution operator  $U^n \simeq e^{-in\tau(2J_x^2 + J_z^2)} = e^{-i\chi(2J_x^2 + J_z^2)t/3}$ , equivalent to that given by an effective Hamiltonian  $H_{\text{eff}} = \chi(2J_x^2 + J_z^2)/3 = \chi(J_x^2 - J_y^2 + \vec{J}^2)/3$ . After dropping a constant  $\vec{J}^2 = j(j+1)$ ,  $H_{\text{eff}}$  reduces to that of the TAT. This shows the application of our proposed  $\pi/2$  pulse sequence effectively transforms the OAT

models of the recent experiments [4, 5] into the TAT models, provided  $\tau \ll (2N)^{-1}$  [26]. For a fixed  $\chi$ , the time of optimal spin squeezing from the TAT  $\propto \chi(J_x^2 - J_y^2)$  is about 1/3 of that from the OAT  $\propto \chi J_z^2$  with  $N = 1250$ , i.e., squeezing occurs around 3 times faster for the TAT. Thus, despite of the three times reduction in the effective strength ( $\chi/3$ ) of the transformed Hamiltonian  $H_{\text{eff}}$ , the time for observing the optimal SS remains almost the same. Consequently, we expect degradation from atomic losses will be similar to the case of OAT [27].

The validity for our idea of transforming the OAT into the TAT can be directly checked through comparing the effective dynamics from  $H_{\text{eff}}$  with the actual dynamics due to (1) accompanied with the sequence of pulses. For this purpose, we expand the state vector time evolved from the initial state  $|j, j\rangle$ ,

$$|\Psi(t)\rangle = e^{-i\chi t(J_x^2 - J_y^2)/3} |j, j\rangle = \sum'_m c_m(t) |j, m\rangle, \quad (4)$$

into eigenstate of  $J_z$  with  $j = N/2$ . The primed summation implies  $m = -j, -j+2, \dots, j$ , or  $m = -j+1, -j+3, \dots, j$ , for even or odd  $N$  respectively [2].

First we consider the case of small  $N$ , e.g.,  $N = 2$ . The quantum dynamics of the effective TAT Hamiltonian  $H_{\text{eff}} = \chi(J_x^2 - J_y^2)/3$  give rise to time evolved probability amplitudes  $c_{-1}(t) = -i\sin(\chi t/3)$ ,  $c_0(t) = 0$ , and  $c_1(t) = \cos(\chi t/3)$ . For each period  $t_c = 3\delta t$  of the exact dynamics given by Eq. (3), the probability amplitudes become  $c_{-1}(t) = e^{-i\chi(t+nt_c)}/2$ ,  $c_0(t) = e^{-i\chi\delta t}/\sqrt{2}$ ,  $c_1(t) = e^{-i\chi(t+nt_c)}/2$  for  $(n-1)t_c < t \leq (n-1)t_c + 2\delta t$ , and  $c_{-1}(t) = -ie^{-i\chi(t-nt_c)/3}\sin(\chi nt_c/3)$ ,  $c_0(t) = 0$ ,  $c_1(t) = e^{-i\chi(t-nt_c)/3}\cos(\chi nt_c/3)$  for  $nt_c - \delta t < t \leq nt_c$ . At  $t = nt_c$ , i.e., after  $n$  periods, they become  $c_{-1}(t) = -ie^{-i2\chi t/3}\sin(\chi t/3)$ ,  $c_0(t) = 0$ , and  $c_1(t) = e^{-i2\chi t/3}\cos(\chi t/3)$ , exactly the same as that from the effective TAT dynamics, apart from an overall phase for the  $\vec{J}^2$  term.

Next we consider larger  $N$  numerically, e.g.,  $N = 1250$  atoms and for different number of pulses  $N_c$  during the time of optimal squeezing  $\chi t \approx 3\ln(4N)/(2N)$  [26]. The TAT Hamiltonian  $H_{\text{eff}}$  can generate optimal spin squeezing when applied to the initial state  $|j, j\rangle = |\uparrow\rangle^{\otimes N}$  [2]. The state expansion Eq. (4) involves basis  $|j, m\rangle$  with even or odd  $m$  because of the conserved parity  $\exp(i\pi J_z)$ . As a result, the mean spin is always along the  $z$ -axis, i.e.,  $\langle \vec{J} \rangle = (0, 0, \langle J_z \rangle)$ . For any spin component normal to  $\langle \vec{J} \rangle$ ,  $J_\gamma = J_x \cos \gamma + J_y \sin \gamma$  with an arbitrary angle  $\gamma$ , its variance is found to be

$$(\Delta J_\gamma)^2 = \frac{1}{2}[\mathcal{C} + \mathcal{A} \cos(2\gamma) + \mathcal{B} \sin(2\gamma)], \quad (5)$$

where  $\mathcal{A} = \langle J_x^2 - J_y^2 \rangle = 3\chi^{-1}\langle H_{\text{eff}} \rangle$ ,  $\mathcal{B} = \langle J_x J_y + J_y J_x \rangle = \text{Im}\langle J_+^2 \rangle$ , and  $\mathcal{C} = \langle J_x^2 + J_y^2 \rangle = j(j+1) - \langle J_z^2 \rangle$ . The optimal squeezing angle  $\gamma_{\text{op}}$  is obtained from minimizing  $(\Delta J_\gamma)^2$  with respect to  $\gamma$ , yielding  $\tan(2\gamma_{\text{op}}) = \mathcal{B}/\mathcal{A}$ . As the coefficient  $\mathcal{A} = \text{Re}\langle J_+^2 \rangle = 0$ , we have  $\gamma_{\text{op}} = \pi/4$  or

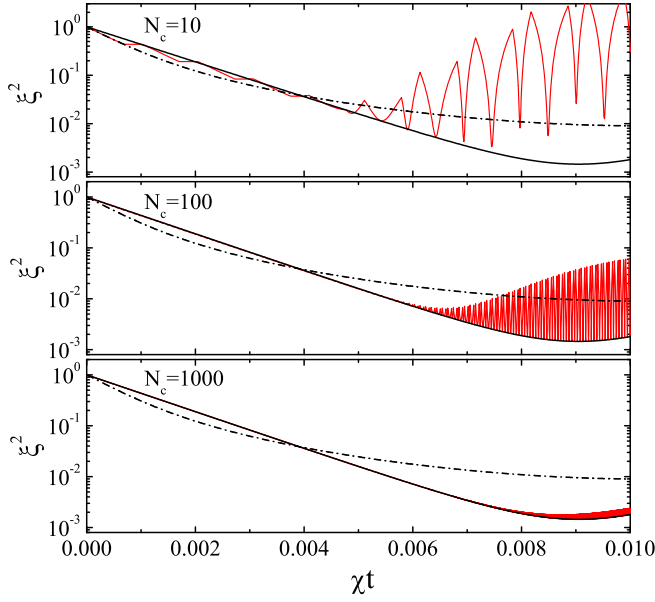


FIG. 2: Spin squeezing parameter at different pulse numbers  $N_c = 10$  (upper), 100 (middle), and 1000 (lower) compared with the effective TAT  $H_{\text{eff}}$  dynamics in black solid lines with the actual ones (1) using the proposed sequence of pulses in red solid lines. The black dot-dashed lines denote OAT results, all for  $N = 1250$  atoms.

$3\pi/4$ , which implies that squeezing and anti-squeezing occurs for  $J_x \pm J_y$ , *i.e.*, along the angle bisector of  $x$ - and  $y$ - axis, with the reduced variance  $V_- = (\mathcal{C} - |\mathcal{B}|)/2$ . Alternatively, the increased variance  $V_+ = (\mathcal{C} + |\mathcal{B}|)/2$  is associated with anti-squeezing.

The reported degree of squeezing is measured by spin squeezing parameter  $\xi_S^2 = NV_-/\langle \vec{J} \rangle^2$  from Ramsey spectroscopy [3], which differs slightly from  $\xi^2 = 2V_-/j$ . The latter form is used to graph spin fluctuations in the relevant figures of Refs. [4, 5], which is independent of angular momentum coordinate system or specific measurement scheme. For the coherent spin state  $|j, j\rangle$ , the variances  $(\Delta J_\gamma)^2 = j/2$  and  $\xi^2 = 1$ . Time evolution from  $H_{\text{eff}}$ , develops quantum correlation, which transforms  $|j, j\rangle$  into a SSS with  $\xi^2 < 1$ . As shown in Fig. 2, with the increase of pulse number  $N_c$ , the actual OAT dynamics approach and eventually settle down to the effective dynamics of  $H_{\text{eff}}$ . The minimum number of periods required before reaching the optimal squeezing point is found analytically and checked numerically to satisfy  $N_c \gg \ln(4N)$  [26] in order to generate the effective dynamics of the TAT. For  $N = 1250$ ,  $\ln(4N) \approx 8.5$ , thus it is reasonable to expect a near perfect agreement at  $N_c = 1000$ . In practice, without special control techniques, errors from repeated pulses can build up, limiting the performance of our proposal.

Fortunately, however, we find that even for significantly fewer number of pulses, the TAT-type spin squeezing limit can still be reached at selected times before ar-

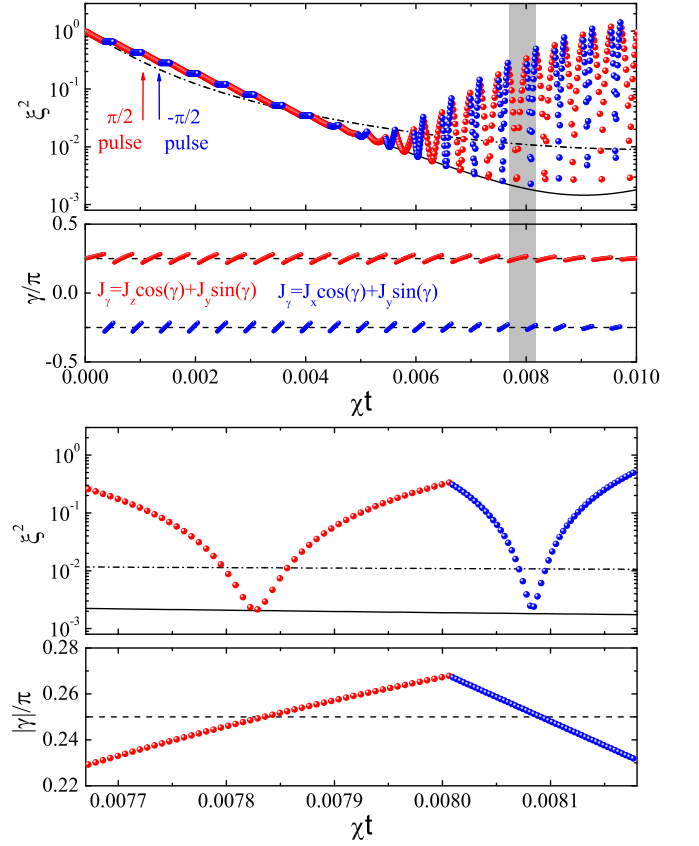


FIG. 3: Top-higher panel: Spin squeezing parameter calculated from the actual dynamics of (1) for  $N_c = 20$  pulses shown by red and blue dots respectively for time windows after the first (red) and second (blue) pulses of each cycle. For comparison, the black solid (dot-dashed) line denotes results from the associated TAT (OAT) model. The top-lower panel shows the corresponding optimal squeezing measurement angle. The dichotomy of two narrow distributions around  $\pm\pi/4$  is due to the repeated  $\pi/2$  pulses rotating between the  $x$ - and  $z$ -axis. For the TAT around the  $z$ - or  $x$ -axis, this angle is simply a flat distribution at  $\pm\pi/4$ , as shown by black dashed lines. At the reduced number of cycles  $N_c = 20$ , the TAT is not yet fully effective, which is the cause for small sloped distributions through  $\pm\pi/4$ . The slopes eventually reduce to zero with the increase of  $N_c$ . The bottom panels display in detail the shaded region where the slopes are now displayed in absolute values, all for  $N = 1250$  atoms.

iving at the optimal squeezing. This is presented in the upper panel of Fig. 3, where spin squeezing obtained from the actual dynamics (1) are shown by red and blue dots respectively for times after the first (red) and second (blue) pulses of each period. One can see clearly that in segments of red and blue dots, the oscillating squeezing parameter kisses the TAT spin squeezing represented by the black solid curve repeatedly. Furthermore, from a technical point of view, the measurement of spin squeezing can be carried out more straightforwardly as the angle for optimal squeezing  $\gamma_{\text{op}}$  is essentially fixed, corresponds to a nearly flat distribution in the immediate temporal

neighborhood as shown in the lower panel of Fig. 3.

The dynamic behavior of spin squeezing from the Hamiltonian (1) has been studied before [24, 25], including the idea of turning off the OAT interaction [28], and the projection into a desired form [29] as in NMR dynamic decoupling. Our idea of multiple pulses as presented in this work is straightforward. It comes with a clear physical picture, and makes use of coherent control techniques [30, 31]. With suitable generalization, our work suggests that for systems of identical spin-1/2 atoms, any forms of binary interactions, e.g., the famous Lipkin-Meshkov-Glick model [32], can be transformed into the effective TAT-type provided their original spin-spin interaction is NOT SU(2) symmetric, or NOT proportional to  $\vec{J}^2 = J_x^2 + J_y^2 + J_z^2$ . We can further adopt techniques like spin echo using additional  $\pi$ -pulses middle way between all neighboring  $\pm\pi/2$ -pulses, to effectively suppress both spin-spin relaxation or other inhomogeneous effects.

Before conclusion, we note that in all our simulations for the actual dynamics of (1), we have assumed the  $\pm\pi/2$  control pulses are executed quickly, during which the nonlinear twisting interaction is negligible. The results presented above in Figs. 2 and 3 are found to remain unchanged irrespective of whether square shaped or Gaussian shaped pulses are used as long as they give the correct pulse area of  $\pi/2$  and are short enough so that spin squeezing during the pulses can be neglected. When the phase accumulated from spin squeezing interactions are included, they generally lead to a gradual degradation of the achievable spin squeezing. Based on numerical simulations over a broad range of parameters [2, 33], we conclude for large  $N$  ( $> 100$ ) the time for optimal squeezing  $\chi t_{\text{op}} \simeq 1.58 \ln(N)/N$  [26], consistent with the earlier result [34].

In conclusion, we present an idea based on coherent control theory. Adopting repeated Rabi pulses we transform the OAT spin squeezing observed in the two recent experiments [4, 5] into stronger and more effective TAT spin squeezing. Both analytical analysis and numerical simulations are presented that confirm the validity of our proposal. The Heisenberg limited noise reduction  $\propto 1/N$  from our proposal is an additional factor  $\propto 1/N^{1/3}$  lower than that reached by the OAT model [2]. This work thus enables improved spin squeezing from  $\propto 1/N^{2/3}$  to  $\propto 1/N$  despite a factor of 1/3 reduction in the effective atomic nonlinear interaction strength. Additionally, the optimal squeezed direction/angle for the TAT interaction is free from the swirling associated with the OAT [2, 4, 5], allowing for simpler and cleaner measurements. Our key idea of controlled dynamics to affect a TAT interaction Hamiltonian from a OAT form can be further generalized to other systems where OAT spin squeezing are discussed [12, 35, 36]. Our proposal calls for no major complications to the available experimental setups, except for a more advanced timing sequence for executing Rabi

pulses while keep tracking of their phases. Thus we fully expect that our idea can be realized within current experiments and hopefully implemented immediately. The experimental demonstration of our proposal could significantly push the frontier of quantum metrology with squeezed spin states into new territory.

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